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A new approach to vector-valued rational interpolation

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Abstract

In this work we propose three different procedures for vector-valued rational interpolation of a function $F(z)$, where $F: \mathbb{C} \to \mathbb{C}^N$, and develop algorithms for constructing the resulting rational functions. We show that these procedures also cover the general case in which some or all points of interpolation coalesce. In particular, we show that, when all the points of interpolation collapse to the same point, the procedures reduce to those presented and analyzed in an earlier paper [J. Approx. Theory 77 (1994) 89] by the author, for vector-valued rational approximations from Maclaurin series of $F(z)$. Determinant representations for the relevant interpolants are also derived.

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1. Introduction

In an earlier work, Sidi [4], we presented three different kinds of vector-valued rational approximations derived from the Maclaurin series $\sum_{i=0}^{\infty} u_i z^i$ of a vector-valued function $F(z)$, where $F: \mathbb{C} \to \mathbb{C}^N$. Here $u_i \in \mathbb{C}^N$ are vectors independent of z. These approximations were based on the minimal polynomial extrapolation (MPE), the modified minimal

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polynomial extrapolation (MMPE), and the topological epsilon algorithm (TEA), three extrapolation methods used for accelerating the convergence of certain kinds of vector sequences; they were shown to have Padé-like algebraic properties and were denoted SMPE, SMMPE, and STEA, respectively, for short. Following their derivation, we also provided in [4] detailed convergence analyses of de Montessus and Koenig types pertaining to the case in which $F(z)$ is analytic at $z = 0$ and meromorphic in some open disk $K = \{z : |z| < R\}$.

The results of [4] show that SMPE, SMMPE, and STEA are effective approximation procedures for such functions $F(z)$. The effectiveness of the procedures SMPE, SMMPE, and STEA is also attested to by their close connection with well-known Krylov subspace methods, such as those of Arnoldi and of Lanczos, for approximating eigenpairs of large sparse matrices. For details, see Sidi [5], where some of the literature on vector extrapolation and Krylov subspace methods is also mentioned.

In the present work, we treat the problem of *interpolating* the function $F(z)$ by vector-valued rational functions along lines similar to those of [4]. We derive three different types of rational interpolation procedures, which we denote IMPE, IMMPE, and ITEA for short, in analogy to SMPE, SMMPE, and STEA, respectively. We show that these procedures remain valid for the general case in which some or all points of interpolation coalesce. In particular, when all points of interpolation collapse to the same point, IMPE, IMMPE, and ITEA reduce to SMPE, SMMPE, and STEA, respectively. This, along with the convergence theory given in [4] and the developments in [5], indicates that the new interpolation procedures of the present paper are likely to have good convergence properties, at least when $F(z)$ is analytic in some bounded open set K_0 and meromorphic in some other open set K_1 whose interior contains K_0.

In addition, in case $N = 1$, the approach we propose here, is designed such that the procedure ITEA produces the solution to the (scalar) Cauchy interpolation problem. This provides another justification of our approach.

In the next section, we give the general framework within which we can define a whole family of vector-valued rational interpolants. The denominators of these interpolants are scalar-valued polynomials whose coefficients can be chosen in different ways. Their numerators are vector-valued polynomials that are constructed to satisfy the interpolation conditions. Of course, for effective approximations, the denominator polynomials need to be constructed carefully according to sensible criteria. This is the subject of Section 3, where we introduce three different types of criteria to obtain the three types of rational interpolation procedures we alluded to above. We emphasize here that, unlike scalar rational interpolation, vector-valued rational interpolation cannot be dealt with only on the basis of interpolation conditions; one needs additional criteria to define the interpolants.

In Section 4, we derive determinantal representations for these rational interpolants. The determinantal representations of Section 4 may serve as a useful tool in the (de Montessus type) convergence analysis of the interpolants as the degree of their numerators tends to infinity while the degree of their denominators is kept fixed. This approach was used successfully in [4] and some of the papers referred to there. We propose to come back to this study in a future publication.

Methods for vector-valued rational interpolation have been considered in the literature. See, for example, Graves-Morris [1,2], Graves-Morris and Jenkins [3], and Van Barel and

Bultheel [6]. To the best of our knowledge, the methods we propose in the present work are different.

2. General approach to vector-valued rational interpolation

Let z be a complex variable and let $F(z)$ be a vector-valued function such that $F : \mathbb{C} \to \mathbb{C}^N$. Assume that $F(z)$ is defined in a bounded open set Ω and consider the problem of interpolating $F(z)$ at some of the points $\xi_1, \xi_2, \ldots,$ in this set. For the moment, we assume that the ξ_i are distinct.

Let $G_{m,n}(z)$ be the vector-valued polynomial (of degree at most $n - m$) that interpolates $F(z)$ at the points $\xi_m, \xi_{m+1}, \ldots, \xi_n$. Thus, in Newtonian form, this polynomial is given as in

$$\begin{aligned} G_{m,n}(z) = F[\xi_m] &+ F[\xi_m, \xi_{m+1}](z - \xi_m) \\ &+ F[\xi_m, \xi_{m+1}, \xi_{m+2}](z - \xi_m)(z - \xi_{m+1}) \\ &+ \cdots + F[\xi_m, \xi_{m+1}, \ldots, \xi_n](z - \xi_m)(z - \xi_{m+1}) \cdots (z - \xi_{n-1}). \end{aligned} \tag{2.1}$$

Here, $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ is the divided difference of order s of $F(z)$ over the set of points $\{\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}\}$. The $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ are defined, as in the scalar case, by the recursion relations

$$\begin{aligned} &F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}] = \frac{F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s-1}] - F[\xi_{r+1}, \xi_{r+2}, \ldots, \xi_{r+s}]}{\xi_r - \xi_{r+s}}, \\ &\quad r = 1, 2, \ldots, \; s = 1, 2, \ldots, \end{aligned} \tag{2.2}$$

with the initial conditions

$$F[\xi_r] = F(\xi_r), \quad r = 1, 2, \ldots . \tag{2.3}$$

Obviously, $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ are all vectors in $\mathbb{C}^N$.

Before we proceed, we would like to emphasize that we employ the representation of the interpolating polynomials via the Newton formula in our work not as a matter of convenience; we make actual use of it in fixing criteria for defining our vector-valued rational approximations.

For simplicity of notation, we define the scalar polynomials $\psi_{m,n}(z)$ via

$$\psi_{m,n}(z) = \prod_{r=m}^{n} (z - \xi_r), \quad n \geqslant m \geqslant 1; \quad \psi_{m,m-1}(z) = 1, \quad m \geqslant 1. \tag{2.4}$$

We also define the vectors $D_{m,n}$ via

$$D_{m,n} = F[\xi_m, \xi_{m+1}, \ldots, \xi_n], \quad n \geqslant m. \tag{2.5}$$

With this notation, we can rewrite (2.1) in the form

$$G_{m,n}(z) = \sum_{i=m}^{n} D_{m,i}\, \psi_{m,i-1}(z). \tag{2.6}$$

We now define a general class of vector-valued rational functions $R_{p,k}(z)$ by

$$R_{p,k}(z) = \frac{U_{p,k}(z)}{V_{p,k}(z)} = \frac{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)\, G_{j+1,p}(z)}{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)}, \tag{2.7}$$

where $c_0, c_1, \ldots, c_k$ are, for the time being, arbitrary complex scalars. Obviously, $U_{p,k}(z)$ is a vector-valued polynomial of degree at most $p-1$ and $V_{p,k}(z)$ is a scalar polynomial of degree at most k. Note that, provided $V_{p,k}(\xi_1) \neq 0$, we can normalize $V_{p,k}(z)$ so that $V_{p,k}(\xi_1) = c_0 = 1$.

The next lemma shows that, when the ξ_i are distinct, $R_{p,k}(z)$ interpolates $F(z)$.

Lemma 2.1. *Assume that the ξ_i are distinct. Provided $V_{p,k}(\xi_i) \neq 0$, $i = 1, 2, \ldots, p$, the vector-valued rational function $R_{p,k}(z)$ interpolates $F(z)$ at the points $\xi_1, \xi_2, \ldots, \xi_p$.*

Proof. First,

$$\begin{aligned} F(z) - R_{p,k}(z) &= \frac{V_{p,k}(z)F(z) - U_{p,k}(z)}{V_{p,k}(z)} \\ &= \frac{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)[F(z) - G_{j+1,p}(z)]}{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)}. \end{aligned} \tag{2.8}$$

Next, letting $z = \xi_i$ in (2.8), and using the fact that

$$G_{j+1,p}(\xi_i) = F(\xi_i), \quad i = j+1, j+2, \ldots, p \tag{2.9}$$

and the fact that

$$\psi_{1,j}(\xi_i) = 0, \quad i = 1, \ldots, j, \tag{2.10}$$

we realize that $\psi_{1,j}(z)[F(z) - G_{j+1,p}(z)] = 0$ for $i = 1, 2, \ldots, p$. This completes the proof. $\square$

In the next lemma, we analyze the limit of $R_{p,k}(z)$ as $\xi_i \to 0$ for all i. The proof of this lemma can be achieved by recalling that, if $F(z)$ has s continuous derivatives at ζ, then

$$\lim_{\substack{\zeta_i \to \zeta \\ i=0,1,\ldots,s}} F[\zeta_0, \zeta_1, \ldots, \zeta_s] = F[\zeta, \zeta, \ldots, \zeta] = \frac{F^{(s)}(\zeta)}{s!}, \quad s = 0, 1, \ldots . \tag{2.11}$$

Lemma 2.2. *Assume that $F(z)$ is differentiable at $z = 0$ as many times as necessary. Letting $\xi_i \to 0$ for all i, we have*

$$\lim_{\substack{\xi_i \to 0 \\ i=1,2,\ldots,p}} R_{p,k}(z) = \frac{\sum_{j=0}^{k} c_j z^j F_{p-j-1}(z)}{\sum_{j=0}^{k} c_j z^j}, \tag{2.12}$$

where

$$F_m(z) = \sum_{i=0}^{m} u_i z^i, \quad m = 0, 1, \ldots; \quad u_i = \frac{F^{(i)}(0)}{i!}, \quad i = 0, 1, \ldots . \tag{2.13}$$

Note that the resulting limit of $R_{p,k}(z)$ in Lemma 2.2 satisfies $F(z) - R_{p,k}(z) = O(z^p)$ as $z \to 0$. It is thus of the form given originally in [4], with $p = n + k$ there.

The next lemma shows that $R_{p,k}(z)$, precisely as defined by (2.7), interpolates $F(z)$ (in the sense of Hermite) also when some points of interpolation coincide. An important point to recall in this connection is that the divided differences $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ are defined via the recursion relation given in (2.2), provided we pass to the limit in case $\xi_r = \xi_{r+s}$ there. The divided difference table for $F(z)$ can be computed very conveniently in this case if we order the points ξ_i as in Lemma 2.3 below and make use of (2.11) when necessary.

Lemma 2.3. *Let $a_1, a_2, \ldots,$ be distinct complex numbers, and let*

$$\begin{aligned} &\xi_1 = \xi_2 = \cdots = \xi_{r_1} = a_1\\ &\xi_{r_1+1} = \xi_{r_1+2} = \cdots = \xi_{r_1+r_2} = a_2\\ &\xi_{r_1+r_2+1} = \xi_{r_1+r_2+2} = \cdots = \xi_{r_1+r_2+r_3} = a_3\\ &\textit{and so on.} \end{aligned} \tag{2.14}$$

Let t and ρ be the unique integers satisfying $t \geqslant 0$ and $0 \leqslant \rho < r_{t+1}$ for which $p = \sum_{i=1}^{t} r_i + \rho$. Then, $R_{p,k}(z)$, as defined in (2.7), *and with $V_{p,k}(a_i) \neq 0$ for all i, interpolates $F(z)$ as follows*:

$$\begin{aligned} R_{p,k}^{(s)}(a_i) = F^{(s)}(a_i), \quad &\textit{for } s = 0, 1, \ldots, r_i - 1 \quad \textit{when } i = 1, \ldots, t,\\ &\textit{and for } s = 0, 1, \ldots, \rho - 1 \quad \textit{when } i = t + 1. \end{aligned} \tag{2.15}$$

(*Of course, when $\rho = 0$, there is no interpolation at a_{t+1}.*)

Proof. We start by recalling that, with $n \geqslant m$, $G_{m,n}(z)$ is the generalized Hermite interpolation polynomial to $F(z)$ at the points $\xi_m, \xi_{m+1}, \ldots, \xi_n$, also when these points are not necessarily distinct. We need to analyze each of the terms $\psi_{1,j}(z)[F(z) - G_{j+1,p}(z)]$ in the numerator of (2.8). For this, it is sufficient to study the term with $0 \leqslant j \leqslant r_1 - 1$ when $r_1 > 1$. The analysis of the rest of the terms is identical. Now, $G_{j+1,p}(z)$ is the vector-valued polynomial that interpolates $F(z)$ at $a_1, a_2, \ldots, a_{t+1}$ as in

$$\begin{aligned} &G_{j+1,p}^{(s)}(a_i) = F^{(s)}(a_i), \quad \text{for } 0 \leqslant s \leqslant r_1 - j - 1 \quad \text{when } i = 1,\\ &\quad \text{for } 0 \leqslant s \leqslant r_i - 1 \quad \text{when } i = 2, \ldots, t, \quad \text{and for } 0 \leqslant s \leqslant \rho - 1\\ &\quad \text{when } i = t + 1. \end{aligned} \tag{2.16}$$

Using this and (2.4), we realize that $\psi_{1,j}(z)[F(z) - G_{j+1,p}(z)]$ vanishes at the points $\xi_1, \xi_2, \ldots, \xi_p$, taking multiplicities into account. That is, for every $j \in \{0, 1, \ldots, k\}$, there holds

$$\left(\frac{d^s}{dz^s}\{\psi_{1,j}(z)[F(z)-G_{j+1,p}(z)]\}\right)\Bigg|_{z=a_i}=0,$$
$$\text{for } 0\leqslant s\leqslant r_i-1 \quad \text{when } 1\leqslant i\leqslant t, \quad \text{and for } 0\leqslant s\leqslant \rho-1 \quad \text{when } i=t+1. \tag{2.17}$$

Taking the sth derivative of both sides of (2.8), and invoking (2.17) with the assumption that $V_{p,k}(a_i)\neq 0$ for all i, we complete the proof. □

3. Choice of the c_j

So far, the c_j in (2.7) are arbitrary. Of course, the quality of $R_{p,k}(z)$ as an approximation to $F(z)$ depends very strongly on the choice of the c_j. Naturally, the c_j must depend on $F(z)$ and on the ξ_i. In this section, we discuss precisely the idea of what may be a good choice of c_j. We are assuming that the ξ_i are not necessarily distinct and are ordered as in Lemma 2.3.

Using the short-hand notation

$$\widehat{D}_{m,n}(z)=F[\xi_m,\xi_{m+1},\ldots,\xi_n,z], \quad n\geqslant m \tag{3.1}$$

and recalling that $G_{m,n}(z)$ is the polynomial that interpolates $F(z)$ at the points $\xi_m,\xi_{m+1},\ldots,\xi_n$, we have the error formula

$$F(z)-G_{m,n}(z)=\widehat{D}_{m,n}(z)\psi_{m,n}(z). \tag{3.2}$$

Consequently, we have for $0\leqslant j\leqslant k$ and $n\geqslant p$,

$$F(z)=G_{j+1,p}(z)+\sum_{s=p+1}^{n}D_{j+1,s}\psi_{j+1,s-1}(z)+\widehat{D}_{j+1,n}(z)\psi_{j+1,n}(z). \tag{3.3}$$

Substituting this expression in (2.8), we obtain

$$F(z)-R_{p,k}(z)=\frac{\Delta_{p,k}(z)}{V_{p,k}(z)}, \tag{3.4}$$

where

$$\begin{aligned}\Delta_{p,k}(z)&=\sum_{j=0}^{k}c_j\,\psi_{1,j}(z)[F(z)-G_{j+1,p}(z)]\\&=\sum_{j=0}^{k}c_j\,\psi_{1,j}(z)\left\{\sum_{s=p+1}^{n}D_{j+1,s}\psi_{j+1,s-1}(z)+\widehat{D}_{j+1,n}(z)\psi_{j+1,n}(z)\right\}.\end{aligned} \tag{3.5}$$

By (2.4), we can rewrite (3.5) in the form

$$\Delta_{p,k}(z) = \prod_{i=1}^{p}(z-\xi_i)\left[\sum_{s=p+1}^{n}\left\{\sum_{j=0}^{k} c_j D_{j+1,s}\right\}\psi_{p+1,s-1}(z) + \left\{\sum_{j=0}^{k} c_j \widehat{D}_{j+1,n}(z)\right\}\psi_{p+1,n}(z)\right]. \tag{3.6}$$

We now choose the c_j such that the term inside the square brackets in (3.6) is "small" in some sense. To this effect, we propose the following three procedures for defining the c_j:

1. The first term of the summation $\sum_{s=p+1}^{n}$ inside the square brackets in (3.6), namely, the $s = p+1$ term, is the vector $\sum_{j=0}^{k} c_j D_{j+1,p+1}$, and we propose to minimize the norm of this vector. Thus, with the normalization $c_0 = 1$, which we assumed earlier, $c_1, \dots, c_k$ are the solution to the problem

$$\min_{c_1,\dots,c_k}\left\|D_{1,p+1} + \sum_{j=1}^{k} c_j D_{j+1,p+1}\right\|, \tag{3.7}$$

   where $\|\cdot\|$ stands for an arbitrary vector norm in $\mathbb{C}^N$. With the l_1- and l_∞-norms, the optimization problem can be solved by using linear programming. With the l_2-norm, it becomes a least-squares problem, which can be solved numerically via standard techniques. Of course, the inner product $(\cdot\,,\cdot)$ that defines the l_2-norm [that is, $\|u\| = \sqrt{(u,u)}$], is not restricted to the standard inner product $(u,v) = u^*v$; it can be given by $(u,v) = u^*Mv$, where M is a hermitian positive definite matrix.
   We denote the resulting rational interpolation procedure IMPE.
2. Again, with the normalization $c_0 = 1$, we propose to determine $c_1, \dots, c_k$ via the solution of the linear system

$$\left(q_i, D_{1,p+1} + \sum_{j=1}^{k} c_j D_{j+1,p+1}\right) = 0, \quad i = 1, \dots, k, \tag{3.8}$$

   where $q_1, \dots, q_k$ are linearly independent vectors in $\mathbb{C}^N$. This amounts to demanding that the projection of the $s = p+1$ term in the summation $\sum_{s=p+1}^{n}$ inside the square brackets in (3.6) unto the subspace spanned by $q_1, \dots, q_k$ vanish. Note that we can choose the vectors $q_1, \dots, q_k$ to be independent of p or to depend on p.
   We denote the resulting rational interpolation procedure IMMPE.
3. Again, with the normalization $c_0 = 1$, we propose to determine $c_1, \dots, c_k$ via the solution of the linear system

$$\left(q, D_{1,s} + \sum_{j=1}^{k} c_j D_{j+1,s}\right) = 0, \quad s = p+1, p+2, \dots, p+k, \tag{3.9}$$

where q is a nonzero vector in $\mathbb{C}^N$. This amounts to demanding that the first k vectors $\sum_{j=0}^{k} c_j D_{j+1,s}$, $p+1 \leqslant s \leqslant p+k$, in the summation $\sum_{s=p+1}^{n}$ inside the square brackets in (3.6) have zero projection along the vector q.

We denote the resulting rational interpolation procedure ITEA.

The choices of the c_j we have proposed here may at first seem to be ad hoc. This is far from being the case, however, and the following lemma provides the justification of these choices.

Lemma 3.1. *When $\xi_i \to 0$ for all i, the rational functions $R_{n+k,k}(z)$ obtained through* IMPE, IMMPE, *and* ITEA *procedures described above reduce precisely to the corresponding rational functions $F_{n,k}(z)$ obtained through* SMPE, SMMPE, *and* STEA, *respectively, described in* [4].

Proof. We already know from (2.4), (2.11), and (2.13) [and in the notation of (2.13)] that, as $\xi_s \to 0$ for all s, $\psi_{m,m+i-1}(z) \to z^i$, $D_{m,m+i} \to u_i$, and $G_{m,m+i}(z) \to F_i(z)$ for all m and i. These imply that the c_j for IMPE, IMMPE, and ITEA satisfy (after letting $\tilde{c}_j = c_{k-j}$, $j = 0, 1, \ldots, k$)

$$\min_{\tilde{c}_0,\ldots,\tilde{c}_{k-1}} \left\| \sum_{j=0}^{k-1} \tilde{c}_j u_{n+j} + u_{n+k} \right\|, \tag{3.10}$$

$$\left(q_i, \sum_{j=0}^{k-1} \tilde{c}_j u_{n+j} + u_{n+k} \right) = 0, \quad i = 1, \ldots, k, \tag{3.11}$$

$$\left(q, \sum_{j=0}^{k-1} \tilde{c}_j u_{n+i+j} + u_{n+i+k} \right) = 0, \quad i = 0, 1, \ldots, k-1, \tag{3.12}$$

respectively. Precisely these are the conditions that define the procedures SMPE, SMMPE, and STEA along with (2.12) and (2.13). □

Another justification of our formulation of $R_{p,k}(z)$ is provided by the next lemma that concerns the scalar case $N = 1$.

Lemma 3.2. *In case $N = 1$, that is, in case $F(z)$ is a scalar function, $R_{p,k}(z)$ in the* ITEA *approach interpolates $F(z)$ at the points ξ_i, $i = 1, 2, \ldots, p+k$, when we take $(q, D_{m,s}) = D_{m,s}$. Thus, $R_{p,k}(z)$ is the solution to the Cauchy–Jacobi interpolation problem in this case.*

Remark. Recall that the numerator and denominator polynomials of $R_{p,k}(z)$ are of degree $p-1$ and k, respectively, which implies that the number of the coefficients to be determined in $R_{p,k}(z)$ is $p+k$. These are determined by the $p+k$ interpolation conditions above.

Proof. In this case, the equations in (3.9) become

$$\sum_{j=0}^{k} c_j D_{j+1,s} = 0, \quad s = p+1, p+2, \ldots, p+k.$$

As a result, (3.6) becomes

$$\begin{aligned}\Delta_{p,k}(z) = \prod_{i=1}^{p+k}(z-\xi_i)\Bigg[&\sum_{s=p+k+1}^{n}\left\{\sum_{j=0}^{k} c_j D_{j+1,s}\right\}\psi_{p+k+1,s-1}(z) \\ &+\left\{\sum_{j=0}^{k} c_j \widehat{D}_{j+1,n}(z)\right\}\psi_{p+k+1,n}(z)\Bigg].\end{aligned}$$

The result now follows as before. □

Before closing this section, we mention that a hybridization of IMMPE and ITEA is also possible; that is, we can define the c_j via the linear systems

$$\begin{aligned}&\left(q_i, D_{1,p+1} + \sum_{j=1}^{k} c_j D_{j+1,p+1}\right) = 0, \quad i = 1, \ldots, \mu, \\ &\left(q, D_{1,p+1+i} + \sum_{j=1}^{k} c_j D_{j+1,p+1+i}\right) = 0, \quad i = 1, \ldots, k-\mu,\end{aligned} \tag{3.13}$$

where $0 < \mu < k$, and $q_1, \ldots, q_\mu$ are linearly independent vectors in $\mathbb{C}^N$.

Finally, we mention that the methods we have proposed for determining the c_j can be extended to the case in which $F(z)$ is such that $F : \mathbb{C} \to \mathbb{B}$, where $\mathbb{B}$ is a general space, exactly as is shown in [4, Section 6]. This amounts to the introduction of the norm defined in $\mathbb{B}$ when the latter is a normed space (for IMPE), and to the introduction of some bounded linear functionals (for IMMPE and ITEA). With these, the determinant representations of the next section remain unchanged as well. We refer the reader to [4] for the details.

4. Determinantal representations

We now show that all the interpolants $R_{p,k}(z)$ we discussed in the preceding section have simple determinantal representations similar to those given in [4] for SMPE, SMMPE, and STEA. We believe that these representations will serve as a useful tool in the convergence analysis of the sequences $\{R_{p,k}(z)\}_{p=0}^{\infty}$ with fixed k, as in [4], for the cases in which $F(z)$ is analytic in the set Ω and meromorphic in a set containing Ω in its interior.

Theorem 4.1. *The rational functions $R_{p,k}(z)$ defined by* IMPE, IMMPE, *and* ITEA *all have determinant representations of the form*

$$R_{p,k}(z) = \frac{\begin{vmatrix} \psi_{1,0}(z)\, G_{1,p}(z) & \psi_{1,1}(z)\, G_{2,p}(z) & \cdots & \psi_{1,k}(z)\, G_{k+1,p}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}{\begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}, \tag{4.1}$$

where

$$u_{i,j} = \begin{cases} (D_{i+1,p+1}\, ,\, D_{j+1,p+1}) & \textit{for}\ \text{IMPE}, \\ (q_i\, ,\, D_{j+1,p+1}) & \textit{for}\ \text{IMMPE}, \\ (q\, ,\, D_{j+1,p+i}) & \textit{for}\ \text{ITEA}. \end{cases} \tag{4.2}$$

Here, the numerator determinant is vector-valued and is defined by its expansion with respect to its first row. That is, if M_j is the cofactor of the term $\psi_{1,j}(z)$ in the denominator determinant, then

$$R_{p,k}(z) = \frac{\sum_{j=0}^{k} M_j \psi_{1,j}(z)\, G_{j+1,p}(z)}{\sum_{j=0}^{k} M_j \psi_{1,j}(z)}. \tag{4.3}$$

All this is valid also when the ξ_i are not necessarily distinct and are ordered as in Lemma 2.3.

Proof. First, note that the c_j for IMPE, by (3.7), satisfy the normal equations

$$\sum_{j=1}^{k} (D_{i+1,p+1}, D_{j+1,p+1}) c_j = -(D_{i+1,p+1}, D_{1,p+1}), \quad i = 1, \ldots, k. \tag{4.4}$$

Next, the c_j for IMMPE, by (3.8), satisfy the equations

$$\sum_{j=1}^{k} (q_i, D_{j+1,p+1}) c_j = -(q_i, D_{1,p+1}), \quad i = 1, \ldots, k. \tag{4.5}$$

Finally, the c_j for ITEA, by (3.9), satisfy the equations

$$\sum_{j=1}^{k} (q, D_{j+1,p+i}) c_j = -(q, D_{1,p+i}), \quad i = 1, 2, \ldots, k. \tag{4.6}$$

Thus, in all cases, the c_j are the solution of the linear systems

$$\sum_{j=1}^{k} u_{i,j} c_j = -u_{i,0}, \quad i = 1, \ldots, k. \tag{4.7}$$

Next, because the M_j are the cofactors of the elements in the first rows of the numerator and denominator determinants, it follows from (4.1) that

$$\sum_{j=0}^{k} u_{i,\,j} M_j = 0, \quad i = 1, \ldots, k.$$

Dividing the ith equality by M_0, and letting $M_j/M_0 = c_j$, $j = 0, 1, \ldots, k$, we see that the equations in (4.7) are satisfied. The result now follows. $\square$

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